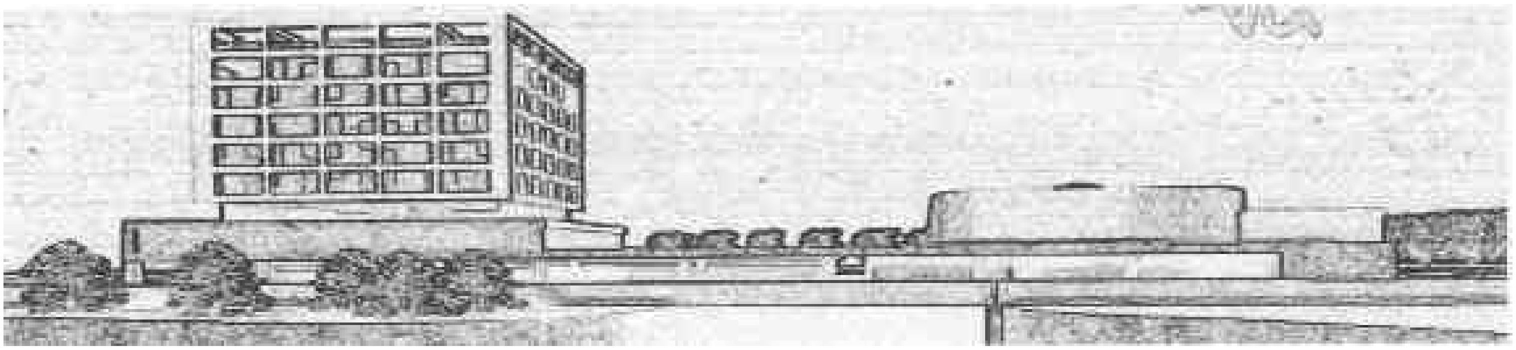
VOLUME X, NUMBER 2



REPORTS ON SYSTEMS AND COMMUNICATIONS

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Móstoles (Madrid), October 2010
Depósito Legal: M-50653-2004
ISSN: 1698-7489

The Length of the Longest Edge in Multi-dimensional Delaunay Graphs *

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Abstract

Upper bounding the length of Delaunay edges in Random Geometric Graphs has been object of recent study in the area of Wireless Networks. In this paper, the problem is generalized to an arbitrary number of dimensions showing upper and lower bounds that hold with parametric probability. The results obtained are asymptotically tight for all relevant values of such probability, and show that the overhead produced by boundary nodes in the plane holds also for more dimensions. To our knowledge, this is the first comprehensive study on this topic.

1 Introduction

The topic of this work is the study of the length of the longest Delaunay edge in multidimensional Euclidean spaces. In particular, the Delaunay graph considered is defined over a set of points distributed uniformly at random in a multidimensional body of unitary volume. The motivation to study such setting comes from the Random Geometric Graph (RGG) model $\mathcal{G}_{n,r}$, where n nodes are distributed uniformly at random in a unit disk or, more generally, according to some specified density function on d-dimensional Euclidean space [9].

It is known [6] that the length of the longest Delaunay edge is strongly influenced by the boundaries of the enclosing body. For instance, if the area enclosing the points is a disk, the longest edge is asymptotically larger than if the area is the surface of a sphere. Therefore, we study the problem for two cases that we call *with boundary* and *without boundary*.

Multidimensional Delaunay tessellations have been studied before with respect to construction algorithmic techniques [5, 7]. Restricted to two dimensions, upper bounding the length of the longest Delaunay edge has attracted interest recently [6] in the context of extensive algorithmic work [1–3] aimed to reduce the energy consumption of geographically routing messages in Radio Networks. The upper bounds presented in [6] are only asymptotical, restricted to $d = 2$, and for

*This research was supported in part by Spanish MICINN grant no. TIN2008-06735-C02-01, Comunidad de Madrid grant no. S2009TIC-1692, EU Marie Curie International Reintegration Grant IRG 210021, and NSF grant no. 0937829.

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enclosing bodies with boundary (although without boundary is implicit because the distance to the boundary is parametric).

The results presented here include upper and lower bounds for d-dimensional bodies with and without boundaries, that hold for a parametric error probability ε. Lower bounds without boundary and all upper bounds apply for any $d > 1$. Lower bounds with boundary are shown for $d \in \{2, 3\}$. The results shown are asymptotically tight for $e^{-cn} \leq \varepsilon \leq n^{-c}$, for any constant $c > 0$. To the best of our knowledge, this is the first comprehensive study of this problem.

In the following section, some necessary notation is introduced. Upper and lower bounds for enclosing bodies without boundaries are enumerated in Section 3, and the case with boundaries is covered in Section 4. We conclude with some open problems. Upper bounds are proved exploiting that, thanks to the uniform density, is very unlikely that a "large" area/volume is void of points. Lower bounds, on the other hand, are proved showing that some configuration that yields a Delaunay edge of certain length is not very unlikely.

2 Preliminaries

The following notation will be used throughout. A *d-sphere*, of radius r is the set of all points in a d-dimensional L_2-space that are located at distance r (called the *radius*) from a given point (called the *center*). A *d-ball*, of radius r is the set of all points in a d-dimensional L_2-space that are located at distance *at most* r (called the *radius*) from a given point (called the *center*). The *area* of a sphere is the area of its surface. The *volume* of a ball is the amount of space it occupies. A *unit sphere* is a sphere of area 1. A *unit ball* is a ball of volume 1.

Let P be a set of points on a d-sphere. Given two points $a, b \in P$, let $\widehat{ab}$ be the arc of a great $(d-1)$-ball intercepted between them. Let $\delta(a, b)$ be the orthodromic distance of such arc. Let the *orthodromic diameter* be the longest orthodromic distance between any pair of points in the surface area of an spherical cap. Let $A_d(x, y)$ be the surface area of a spherical cap of orthodromic diameter y, of a d-sphere of surface area x. Let $V_d(x, y)$ be the volume of a spherical cap of base diameter y, of a d-ball of volume x. Let $D(P)$ be the Delaunay graph of a set of points P.

The following definitions of a Delaunay graph of a set of points in d-dimensional bodies can be derived as in Theorem 9.6.ii in [4].

Definition 1. *Let P be a set of points in a d-sphere, two points $a, b \in P$ form an arc of $D(P)$, if and only if there is a d-dimensional spherical cap C such that, with respect to the surface of the cap, it contains a and b on the boundary and does not contain any other point of P.*

Definition 2. *Let P be a set of points in a d-ball, two points $a, b \in P$ form an edge of $D(P)$, if and only if there is a d-ball B such that, a and b are located in the surface area of B, and the interior of B does not contain any other point of P.*

The following bound [8] is used throughout

$$e^{-x/(1-x)} \leq 1 - x \leq e^{-x}, \text{ for } 0 < x < 1. \tag{1}$$

3 Enclosing Body without Boundary

The following theorems show upper and lower bounds on the length of arcs in the Delaunay graph on a d-sphere.

3.1 Upper Bound

Theorem 3. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit d-sphere, with probability at least $1-\varepsilon$, for $0 < \varepsilon < 1$, there is no arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$A_d(1, \delta(a,b)) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Proof. Let $a, b \in P$ be any pair of points from P separated by an orthodromic distance $\delta(a,b)$. By Definition 1, for the arc $\widehat{ab}$ to be in $D(P)$, there must exist a d-dimensional spherical cap C such that a and b are located on the boundary of the cap base and the cap surface of C is void of points from P. In order to upper bound $Pr(\widehat{ab} \in D(P))$, we upper bound the probability that the surface of C is empty. In order to do that, we lower bound the surface area of C. Consider an spherical cap C' of the unit sphere with orthodromic diameter $\widehat{ab}$. The surface area of C is at least the surface area of C'. Therefore, given that the points are distributed uniformly and independently at random, it is $Pr(\widehat{ab} \in D(P)) \leq (1 - A_d(1, \delta(a,b)))^{n-2}$.

Given that there are $\binom{n}{2}$ pairs of points, using the union bound, we find a $A_d(1, \delta(a,b))$ that yields a probability at most ε of having some arc $\widehat{ab} \in D(P)$, by making

$$\binom{n}{2} (1 - A_d(1, \delta(a,b)))^{n-2} \leq \varepsilon.$$

Then, given that $a \neq b$, it holds that $A_d(1, \delta(a,b)) < 1$. Then, using Inequality 1, it is enough

$$\exp\left(-A_d(1, \delta(a,b))(n-2)\right) \leq \varepsilon / \binom{n}{2}$$
$$A_d(1, \delta(a,b)) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

□

The following corollaries for $d = 2$ and $d = 3$ can be obtained from Theorem 3 using the corresponding surface areas.

Corollary 4. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit circumference (2-sphere), with probability at least $1-\varepsilon$, for $0 < \varepsilon < 1$, there is no arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Corollary 5. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit sphere (3-sphere), with probability at least $1-\varepsilon$, for $0 < \varepsilon < 1$, there is no arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \geq \frac{1}{\sqrt{\pi}} \arccos\left(1 - \frac{2\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}\right).$$

Proof. The surface area of a spherical cap of a 3-sphere is $2\pi Rh$, where R is the radius of the sphere and h is the height of the cap. For a unit 3-sphere is $R = 1/(2\sqrt{\pi})$. Then, the perimeter of a great 2-ball (the circumference of a great circle) is $2\pi/(2\sqrt{\pi}) = \sqrt{\pi}$. Thus, the central angle of a cap whose orthodromic diameter is ρ is $2\pi\rho/\sqrt{\pi} = 2\sqrt{\pi}\rho$. Let the angle between the line segment $\overline{ab}$ and the radius of the sphere be α. Then,

$$\alpha = \begin{cases} \pi/2 - \sqrt{\pi}\rho & \text{if } \rho \leq \sqrt{\pi}/2 \\ \sqrt{\pi}\rho - \pi/2 & \text{if } \rho > \sqrt{\pi}/2 \end{cases}$$

And the height of the cap is $h = 1/(2\sqrt{\pi}) - 1/(2\sqrt{\pi})\sin(\pi/2 - \sqrt{\pi}\rho) = (1 - \cos(\sqrt{\pi}\rho))/(2\sqrt{\pi})$. Therefore, the surface area of a spherical cap of a 3-sphere whose orthodromic diameter is ρ is $(1 - \cos(\sqrt{\pi}\rho))/2$. Replacing in Theorem 3, the claim follows. □

3.2 Lower Bound

Theorem 6. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit d-sphere, with probability at least ε, there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that $A_d(1, \delta(a, b)) \geq A_d(1, \rho_1)$, where*

$$A_d(1, \rho_1) = \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n - 2 + \ln\left((e-1)/(e^2\varepsilon)\right)},$$

for any $0 < \varepsilon < 1$ such that $A_d(1, 2\rho_1) \leq 1 - 1/(n-1)$.

Proof. For any pair of points $a, b \in P$, by Definition 1, for the arc $\widehat{ab}$ to be in $D(P)$, there must exist a d-dimensional spherical cap C such that a and b are located on the boundary of the cap base and the cap surface of C is void of points from P. We compute the probability of such event as follows.

Let $\rho_2 > \rho_1$ be such that $A_d(1, 2\rho_2) - A_d(1, 2\rho_1) = 1/(n-1)$. Consider any point $a \in P$. The probability that some other point b is located so that $\rho_1 < \delta(a, b) \leq \rho_2$ is

$$1 - \left(1 - \frac{1}{n-1}\right)^{n-1} \geq 1 - 1/e, \text{ by Inequality 1.}$$

The spherical cap with orthodromic diameter $\delta(a, b)$ is empty with probability $(1 - A_d(1, \delta(a, b)))^{n-2}$. To lower bound this probability we consider separately the spherical cap with orthodromic diameter ρ_1 and the remaining annulus of the spherical cap with orthodromic diameter $\delta(a, b)$. The probability that the latter is empty is lower bounded by upper bounding the area $A_d(1, \delta(a, b)) - A_d(1, \rho_1) \leq A_d(1, 2\rho_2) - A_d(1, 2\rho_1) = 1/(n-1)$. Then,

$$\left(1 - \frac{1}{n-1}\right)^{n-2} \geq 1/e, \text{ by Inequality 1.}$$

Finally, the probability that the spherical cap with orthodromic diameter ρ_1 is empty is

$$\begin{aligned}(1 - A_d(1, \rho_1))^{n-2} &\geq \exp\left(-\frac{A_d(1, \rho_1)(n-2)}{1 - A_d(1, \rho_1)}\right), \text{ by Inequality 1} \\ &= \exp\left(-\ln\left(\frac{e-1}{e^2\varepsilon}\right)\right) \\ &= \frac{e^2\varepsilon}{e-1}.\end{aligned}$$

Therefore,

$$Pr\left(\widehat{ab} \in D(P)\right) \geq \left(1 - \frac{1}{e}\right) \frac{1}{e} \frac{e^2 \varepsilon}{e-1} = \varepsilon.$$

□

The following corollaries for $d = 2$ and $d = 3$ can be obtained from Theorem 6 using the corresponding surface areas.

Corollary 7. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit circumference (2-sphere), with probability at least ε, for any $e^{1-n-4/n} \leq \varepsilon < 1$, there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \geq \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n - 2 + \ln\left((e-1)/(e^2\varepsilon)\right)}.$$

Corollary 8. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit sphere (3-sphere), with probability at least ε, for any $e^{-n+2\sqrt{n-1}-1} \leq \varepsilon < 1$, there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \geq \frac{1}{\sqrt{\pi}} \arccos\left(1 - \frac{2\ln\left((e-1)/(e^2\varepsilon)\right)}{n - 2 + \ln\left((e-1)/(e^2\varepsilon)\right)}\right).$$

Proof. As shown in the proof of Corollary 5, the surface area of a spherical cap of a 3-sphere whose orthodromic diameter is ρ is $(1 - \cos(\sqrt{\pi}\rho))/2$. Replacing in Theorem 6, the claim follows. □

4 Enclosing Body with Boundary

The following theorems show upper and lower bounds on the length of edges in the Delaunay graph on a d-ball.

4.1 Upper Bound

Theorem 9. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit d-ball, with probability at least $1 - \varepsilon$, for $0 < \varepsilon < 1$, there is no edge $(a, b) \in D(P)$, $a, b \in P$, such that*

$$V_d(1, ||\overrightarrow{a,b}||_2) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Proof. Let $a, b \in P$ be any pair of points from P. By Definition 2, for the edge (a, b) to be in $D(P)$, there must exist a d-dimensional ball B such that a and b are located on the surface area of B and the interior of B is void of points from P. Notice that B may be such that part of it is outside the unit ball but, given that points are distributed in the unit ball, no point is located outside of it. Then, in order to upper bound $Pr((a, b) \in D(P))$, we upper bound the probability that the interior of the intersection of B with the unit ball is empty. In order to do that, we lower bound the volume of such intersection. For a given distance $||\overrightarrow{a,b}||_2$ such volume is minimized when a

and b are located on the surface of the unit ball, B has infinite radius, and the maximum distance between any pair of points in the intersection between the surface areas of B and the unit ball is $||\overrightarrow{a,b}||_2$. In other words, when the intersection is an spherical cap of the unit ball with base diameter $||\overrightarrow{a,b}||_2$. Therefore, given that the points are distributed uniformly and independently at random, it is $Pr((a,b) \in D(P)) \leq (1 - V_d(1, ||\overrightarrow{a,b}||_2))^{n-2}$.

Given that there are $\binom{n}{2}$ pairs of points, using the union bound, we find a $V_d(1, ||\overrightarrow{a,b}||_2)$ that yields a probability at most ε of having some edge $(a,b) \in D(P)$, by making

$$\binom{n}{2}\left(1 - V_d(1, ||\overrightarrow{a,b}||_2)\right)^{n-2} \leq \varepsilon.$$

Then, given that $a \neq b$, it holds that $V_d(1, ||\overrightarrow{a,b}||_2) < 1$. Then, using Inequality 1, it is enough

$$\begin{aligned} \exp\left(-V_d(1, ||\overrightarrow{a,b}||_2)(n-2)\right) &\leq \varepsilon / \binom{n}{2} \\ V_d(1, ||\overrightarrow{a,b}||_2) &\geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}. \end{aligned}$$

□

The following corollaries for $d = 2$ and $d = 3$ can be obtained from Theorem 9 using the corresponding surface areas.

Corollary 10. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit disk (2-ball), with probability at least $1-\varepsilon$, for $\binom{n}{2}e^{-\sqrt{2}(n-2)/\pi} < \varepsilon < 1$, there is no edge $(a,b) \in D(P)$, $a, b \in P$, such that*

$$||\overrightarrow{a,b}||_2 \geq \sqrt[3]{\frac{16}{\sqrt{\pi}} \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}}.$$

Proof. Consider the intersection of the radius of the unit disk perpendicular to (a,b) with the circumference of the unit disk, call this point d. The area of the triangle $\triangle abd$ is a strict lower bound on $V_2(1, ||\overrightarrow{a,b}||_2)$. From Theorem 9, we have the condition

$$V_2(1, ||\overrightarrow{a,b}||_2) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Thus, it is enough

$$\frac{||\overrightarrow{a,b}||_2}{2}\left(\frac{1}{\sqrt{\pi}} - \sqrt{\frac{1}{\pi} - \frac{||\overrightarrow{a,b}||_2^2}{4}}\right) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Making $\rho = ||\overrightarrow{a,b}||_2\sqrt{\pi}/2$, we want

$$\sqrt{\rho^2 - \rho^4} \leq \rho - \pi\frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}. \tag{2}$$

If $||\overrightarrow{a,b}||_2 < 2\sqrt{\pi}\ln\left(\binom{n}{2}/\varepsilon\right)/(n-2)$, there is nothing to prove because

$$\frac{2\sqrt{\pi}\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2} < \sqrt[3]{\frac{16\ln\left(\binom{n}{2}/\varepsilon\right)}{\sqrt{\pi}(n-2)}},$$

for any $\varepsilon > \binom{n}{2}\exp\left(-\sqrt{2}(n-2)/\pi\right)$. Then, we can square both sides of Inequality 2 getting

$$\rho^4 \geq 2\rho\pi\frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2} - \left(\pi\frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}\right)^2$$
$$\rho^3 \geq 2\pi\frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

And replacing back ρ the claim follows. □

Corollary 11. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit ball (3-ball), with probability at least $1-\varepsilon$, for $\binom{n}{2}e^{-2(n-2)/(3\sqrt{\pi})} < \varepsilon < 1$, there is no edge $(a,b) \in D(P)$, $a, b \in P$, such that*

$$||\overrightarrow{a,b}||_2 \geq \sqrt[4]{\frac{96}{\pi^{3/2}}\frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}}.$$

Proof. Consider the intersection of the radius of the unit ball perpendicular to (a,b) with the surface of the unit ball, call this point d. The volume of the cone whose base is the disk whose diameter is (a,b) and its vertex is d is a strict lower bound on $V_2(1, ||\overrightarrow{a,b}||_2)$. From Theorem 9, we have the condition

$$V_3(1, ||\overrightarrow{a,b}||_2) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Thus, it is enough

$$\frac{\pi}{3}\left(\frac{||\overrightarrow{a,b}||_2}{2}\right)^2\left(\frac{1}{\sqrt{\pi}} - \sqrt{\frac{1}{\pi} - \frac{||\overrightarrow{a,b}||_2^2}{4}}\right) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Making $\rho = ||\overrightarrow{a,b}||_2\sqrt{\pi}/2$, we want

$$\sqrt{\rho^4 - \rho^6} \leq \rho^2 - 3\sqrt{\pi}\frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}. \tag{3}$$

If $||\overrightarrow{a,b}||_2 < \sqrt{12\ln\left(\binom{n}{2}/\varepsilon\right)/(\sqrt{\pi}(n-2))}$, there is nothing to prove because

$$\sqrt{\frac{12\ln\left(\binom{n}{2}/\varepsilon\right)}{\sqrt{\pi}(n-2)}} < \sqrt[4]{\frac{96}{\pi^{3/2}}\frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}},$$

for any $\varepsilon > \binom{n}{2} \exp\left(-2(n-2)/(3\sqrt{\pi})\right)$. Then, we can square both sides of Inequality 3 getting

$$\rho^6 \geq 6\rho^2\sqrt{\pi}\frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2} - \left(3\sqrt{\pi}\frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}\right)^2$$
$$\rho^4 \geq 6\sqrt{\pi}\frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

And replacing back ρ the claim follows. □

4.2 Lower Bound

Theorem 12. *For $d = 2$, given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit d-ball, with probability at least ε, there is an edge $(a, b) \in D(P)$, $a, b \in P$, such that $||\overrightarrow{ab}||_2 \geq \rho_1/2$, where*

$$V_d(1, \rho_1) = \frac{\ln(\alpha/\varepsilon)}{(n - 2 + \ln(\alpha/\varepsilon))},$$

where $\alpha = (1 - e^{-1/16})(1 - e^{-1/32})e^{-1}$, for any $0 < \varepsilon \leq \alpha/e^2$ such that $V_d(1, \rho_1) \leq 1/2 - 1/n$.

Proof. For any pair of points $a, b \in P$, by Definition 2, for the edge (a, b) to be in $D(P)$, there must exist a d-ball such that a and b are located in the surface area of the ball and the interior is void of points from P. We compute the probability of such event as follows. (Refer to Figure 1.) Consider two spherical caps of the unit ball with concentric surface areas, call them Γ_1 and Γ_2, of diameters ρ_1 as defined and ρ_2 such that $V_d(1, \rho_2) = V_d(1, \rho_1) + 1/n$. Let $\Gamma_2 - \Gamma_1$ be all space points in Γ_2 that are not in Γ_1 (i.e. the body defined by the difference of both spherical caps). Inside $\Gamma_2 - \Gamma_1$ (see Figure 1(a)) consider two bodies B_a and B_b of identical volumes such that for any pair of points $a \in B_a$ and $b \in B_b$ the following holds: *(i)* the points a and b are separated a distance at least $\rho_1/2$; *(ii)* there exists an spherical cap Γ containing the points a and b in its base of diameter ρ such that $V_d(1, \rho) \leq V_d(1, \rho_2)$. (See Figure 1(b).) Such event implies the existence of an empty d-ball of infinite radius with a and b in its surface which proves the claim. In the following, we show that such event occurs with big enough probability.

To bound the volume of B_a (hence, B_b), we first bound the ratio ρ_2/ρ_1. Consider the inscribed polygons illustrated in Figure 1(c). It can be seen that the triangle $x_1x_3x_5$ is located inside the pentagon $x_1x_2x_3x_4x_5$ which in turn is composed by the triangle $x_2x_3x_4$ and the trapezoid $x_1x_2x_4x_5$. Then,

$$\frac{(h_1 + h)\rho_2}{2} \leq \frac{\rho_1 h_1}{2} + \frac{(\rho_1 + \rho_2)h}{2}$$
$$h_1\rho_2 \leq (h_1 + h)\rho_1. \tag{4}$$

Given that $\varepsilon \leq \alpha/e^2$, we know that $V_d(1, \rho_1) \geq 2/n$. Then, it holds that $h \leq h_1$. Replacing in Equation 4 we obtain $\rho_2 \leq 2\rho_1$.

Then, the volume of B_a is (see Figure 1(d))

$$\frac{1}{2n} - \frac{\rho_1}{4}h - \left(\frac{\rho_2}{2} - \frac{\rho_1}{4}\right)\frac{h}{2} = \frac{1}{2n} - \frac{\rho_1 + \rho_2}{2}\frac{h}{2} + \frac{\rho_1}{8}h.$$

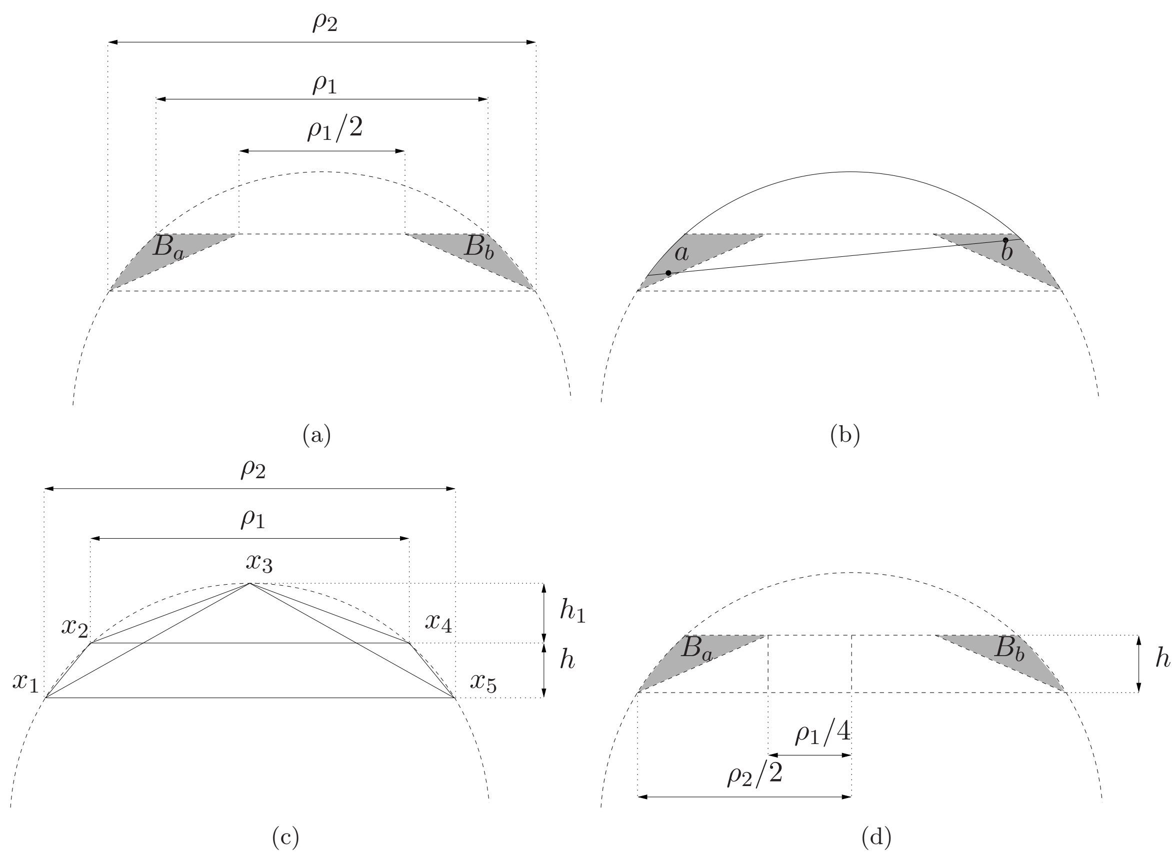

Figure 1: Illustration of Theorem 12.

Using that the volume of the trapezoid $x_1x_2x_4x_5$ is $(\rho_2+\rho_1)h/2 \leq 1/n$, the right-hand side of the latter equation is at least $\rho_1 h/8$ which, using that $\rho_1 \geq \rho_2/2$, is at least $\rho_2 h/16 \geq 1/(16n)$.

Then, the probability that there is a point $a \in P$ located in B_a is

$$> 1 - \left(1 - \frac{1}{16n}\right)^n \geq 1 - e^{-1/16}, \text{ by Inequality 1.}$$

And the probability that there is another point $b \in P$ located in B_b is

$$\begin{aligned} 1 - \left(1 - \frac{1}{16n}\right)^{n-1} &\geq 1 - e^{-(n-1)/(16n)}, \text{ by Inequality 1,} \\ &\geq 1 - e^{-1/32}, \text{ for any } n > 1. \end{aligned}$$

It remains to be shown that Γ is void of points. The probability that Γ is empty is lower bounded by upper bounding the volume, i.e. taking $V_d(1,\rho) \leq V_d(1,\rho_2) = V_d(1,\rho_1) + (V_d(1,\rho_2) - V_d(1,\rho_1))$. For $V_d(1,\rho_2) - V_d(1,\rho_1) = 1/n$, we have

$$\begin{aligned} \left(1 - \frac{1}{n}\right)^{n-2} &\geq e^{-(n-2)/(n-1)}, \text{ by Inequality 1,} \\ &\geq 1/e. \end{aligned}$$

And the probability that Γ_1 is empty is

$$(1 - V_d(1,\rho_1))^{n-2} \geq \exp\left(-\frac{V_d(1,\rho_1)(n-2)}{1 - V_d(1,\rho_1)}\right), \text{ by Inequality 1.}$$

Replacing, we get

$$\begin{aligned} Pr\left((a,b) \in D(P)\right) &\geq \left(1 - \frac{1}{e^{1/16}}\right)\left(1 - \frac{1}{e^{1/32}}\right)\frac{1}{e}\exp\left(-\frac{V_d(1,\rho_1)(n-2)}{1 - V_d(1,\rho_1)}\right) \\ &= \varepsilon. \end{aligned}$$

□

Theorem 13. *For $d = 3$, given the Delaunay graph $D(P)$ of a set P of $n > 4$ points distributed uniformly and independently at random in a unit d-ball, with probability at least ε, there is an edge $(a,b) \in D(P)$, $a, b \in P$, such that $||\overrightarrow{ab}||_2 \geq \rho_1/2$, where*

$$V_d(1,\rho_1) = \frac{\ln(\alpha/\varepsilon)}{(n - 2 + \ln(\alpha/\varepsilon))},$$

where $\alpha = (1 - e^{-1/6})(1 - e^{-1/12})e^{-12}$, for any $0 < \varepsilon \leq \alpha/e$ such that $V_d(1,\rho_1) \leq 1/2 - 1/n$.

Proof. For any pair of points $a, b \in P$, by Definition 2, for the edge (a,b) to be in $D(P)$, there must exist a d-ball such that a and b are located in the surface area of the ball and the interior is void of points from P. We compute the probability of such event as follows. (Refer to the two-dimensional projections of Figure 2.)

Consider two spherical caps of the unit ball with concentric surface areas, call them Γ_1 and Γ_2, of base diameters ρ_1 and ρ_2, and heights h_1 and h_2 respectively. Let ρ_1 be such that $V_d(1,\rho_1)$ is

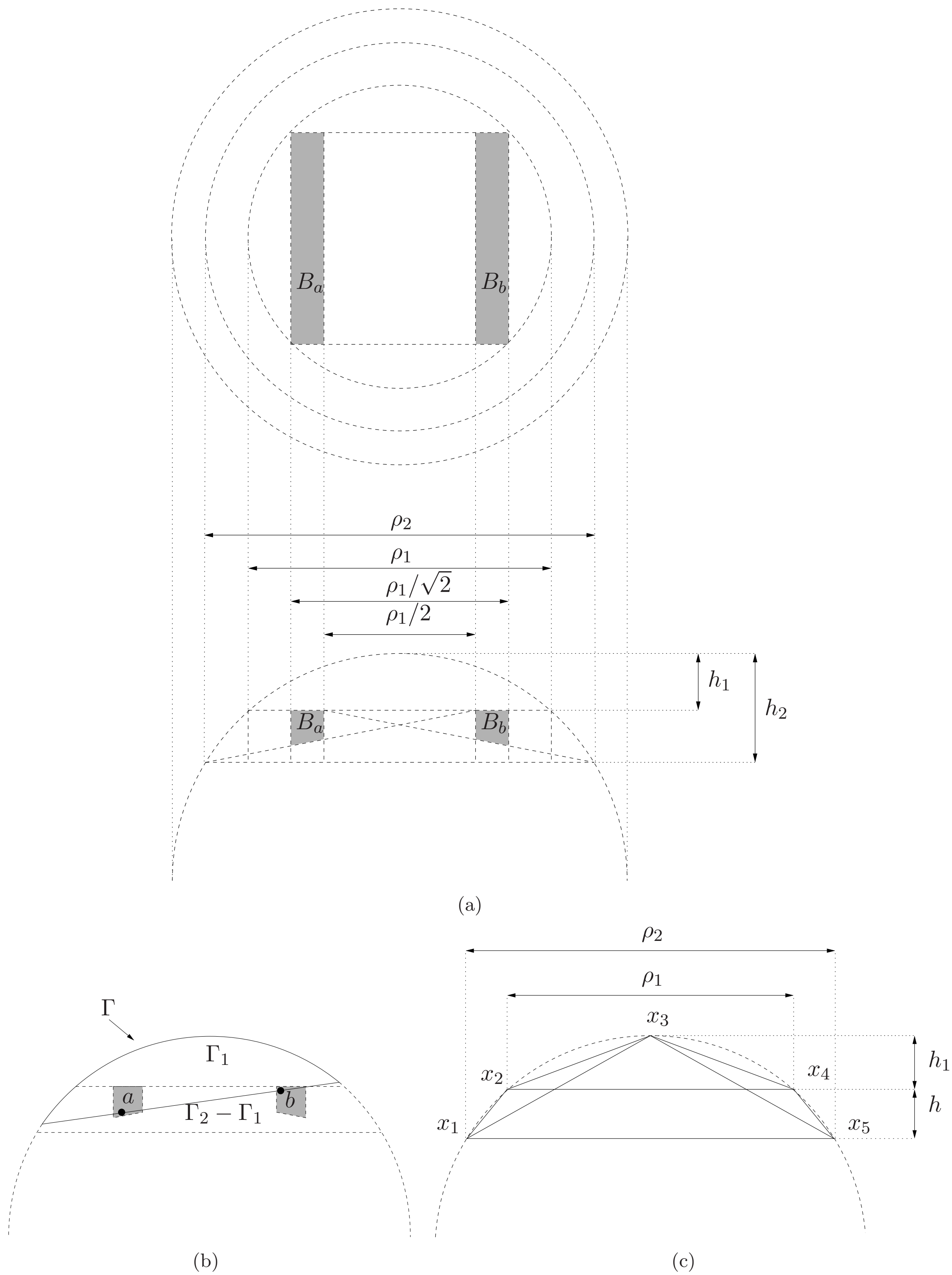

Figure 2: Illustration of Theorem 13.

as defined and let h_2 be such that $\pi(\rho_1/2)^2(h_2-h_1)=1/n$. Let $\Gamma_2-\Gamma_1$ be all space points in Γ_2 that are not in Γ_1 (i.e., the body defined by the difference of both spherical caps). Consider the parallelepiped of sides $\rho_1/\sqrt{2}\times\rho_1/\sqrt{2}\times h_2-h_1$ inscribed in $\Gamma_2-\Gamma_1$ (see Figure 2(a)), call it Π.

Inside Π, consider two bodies B_a and B_b of identical volumes such that for any pair of points $a\in B_a$ and $b\in B_b$ the following holds: *(i)* the points a and b are separated a distance at least $\rho_1/2$; *(ii)* there exists an empty spherical cap Γ that contains the points a and b in its base of diameter ρ such that $V_d(1,\rho)\le V_d(1,\rho_2)$. (See Figure 2(b).) Such configuration implies the existence of an empty d-ball of infinite radius with a and b in its surface which proves the claim. In the following, we show that such configuration occurs with big enough probability.

To bound the volume of B_a (hence, B_b), we first bound the ratio ρ_2/ρ_1. Consider the inscribed bodies whose projection is illustrated in Figure 2(c). It can be seen that the cone $x_1x_3x_5$ is located inside the body composed by the cone $x_2x_3x_4$ and the frustum $x_1x_2x_4x_5$. Then,

$$\begin{aligned}\frac{h_2\pi(\rho_2/2)^2}{d} &\le \frac{h_1\pi(\rho_1/2)^2}{d}+\frac{\pi(\rho_2/2)^3-\pi(\rho_1/2)^3}{d(\rho_2/2-\rho_1/2)}(h_2-h_1)\\ h_1\rho_2 &\le h_2\rho_1. \end{aligned}\tag{5}$$

Given that $\varepsilon\le\alpha/e$, we know that $V_d(1,\rho_1)\ge 1/n$. Then, given that $\pi(\rho_1/2)^2(h_2-h_1)=1/n$, it holds that $h_2\le 2h_1$. Replacing in Equation 5 we obtain $\rho_2\le 2\rho_1$. The base of the big triangle is $\rho_2/2+\rho_1/4$, and the height is h. The base of the triangle to compute is $\rho_1/(2\sqrt{2})+\rho_1/4$, and the height is $h(\rho_1/(2\sqrt{2})+\rho_1/4)/(\rho_2/2+\rho_1/4)$. The base of the small triangle to substract is $\rho_1/2$, and the height is $h\rho_1/(2(\rho_2/2+\rho_1/4))$. Then, the trapezoid area is

$$\begin{aligned}\frac{3}{8}\rho_1 h\frac{\rho_1/(2\sqrt{2})+\rho_1/4}{\rho_2/2+\rho_1/4}-\frac{\rho_1}{4}h\frac{\rho_1}{2(\rho_2/2+\rho_1/4)} &= \frac{\rho_1^2}{2\rho_2+\rho_1}h\left(\frac{3}{2}\left(\frac{1}{2\sqrt{2}}+\frac{1}{4}\right)-\frac{1}{2}\right)\\ &\ge \rho_1 h\frac{1}{4}\left(\frac{3}{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{2}\right)-1\right).\end{aligned}$$

Then, the volume of B_a is at least

$$\begin{aligned}\rho_1^2 h\frac{1}{4\sqrt{2}}\left(\frac{3}{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{2}\right)-1\right) &= \frac{1}{\pi\sqrt{2}n}\left(\frac{3}{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{2}\right)-1\right)\\ &\ge \frac{1}{6n}.\end{aligned}$$

Then, the probability that there is a point $a\in P$ located in B_a is

$$>1-\left(1-\frac{1}{6n}\right)^n\ge 1-e^{-1/6},\ \text{by Inequality 1.}$$

And the probability that there is another point $b\in P$ located in B_b is

$$\begin{aligned}1-\left(1-\frac{1}{6n}\right)^{n-1} &\ge 1-e^{-(n-1)/(6n)},\ \text{by Inequality 1,}\\ &\ge 1-e^{-1/12},\ \text{for any } n>1.\end{aligned}$$

It remains to be shown that Γ is void of points. The probability that Γ is empty is lower bounded by upper bounding the volume, i.e. taking $V_d(1,\rho)\le V_d(1,\rho_2)\le V_d(1,2\rho_1)+(V_d(1,\rho_2)-V_d(1,\rho_1))$.

We know that $V_d(1,\rho_2) - V_d(1,\rho_1) \le \pi(\rho_2/2)^2 h \le \pi(\rho_1)^2 h = 4/n$. Then, for $V_d(1,\rho_2) - V_d(1,\rho_1)$, we have

$$\left(1 - \frac{4}{n}\right)^{n-2} \ge e^{-4(n-2)/(n-4)}, \text{ by Inequality 1,}$$
$$\ge 1/e^{12}, \text{ for any } n > 4.$$

And the probability that Γ_1 is empty is

$$(1 - V_d(1,\rho_1))^{n-2} \ge \exp\left(-\frac{V_d(1,\rho_1)(n-2)}{1 - V_d(1,\rho_1)}\right), \text{ by Inequality 1.}$$

Replacing, we get

$$Pr\left((a,b) \in D(P)\right) \ge \left(1 - \frac{1}{e^{1/6}}\right)\left(1 - \frac{1}{e^{1/12}}\right)\frac{1}{e^{12}}\exp\left(-\frac{V_d(1,\rho_1)(n-2)}{1 - V_d(1,\rho_1)}\right)$$
$$= \varepsilon.$$

□

5 Open Problems

It would be interesting to extend this study to other norms, such as L_1 or L_∞. Also, Theorems 12 and 13 were proved showing that some configuration that yields a Delaunay edge of some length is not unlikely. Different configurations were used for each, but a configuration that works for both cases exists (although yielding worse constants). We conjecture that (modulo some constant) the same bound can be obtained in general for any $d > 1$. Both questions are left for future work.

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