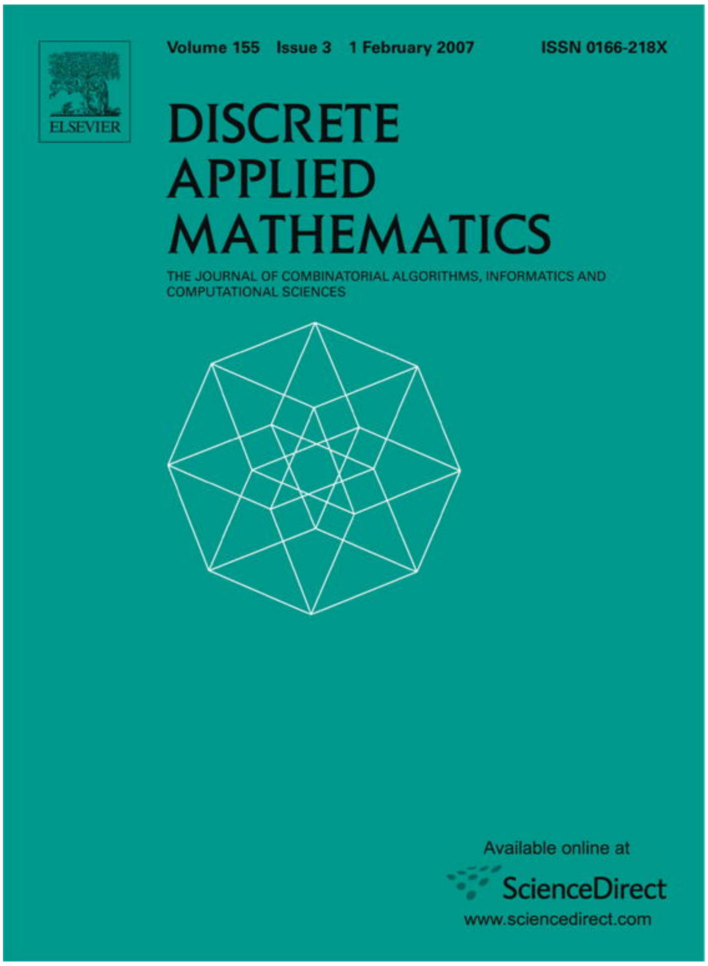
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Containment properties of product and power graphs

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Abstract

In this paper, we study containment properties of graphs in relation with the Cartesian product operation. These results can be used to derive embedding results for interconnection networks for parallel architectures.

First, we show that the isomorphism of two Cartesian powers G^r and H^r implies the isomorphism of G and H, while $G^r \subseteq H^r$ does not imply $G \subseteq H$, even for the special cases when G and H are prime, and when they are connected and have the same number of nodes at the same time.

Then, we find a simple sufficient condition under which the containment of products implies the containment of the factors: if $\prod_{i=1}^{n} G_i \subseteq \prod_{j=1}^{n} H_j$, where all graphs G_i are connected and no graph H_j has 4-cycles, then each G_i is a subgraph of a different graph H_j. Hence, if G is connected and H has no 4-cycles, then $G^r \subseteq H^r$ implies $G \subseteq H$.

Finally, we focus on the particular case of products of graphs with the linear array. We show that the fact that $G \times L_n \subseteq H \times L_n$ does not imply that $G \subseteq H$ even in the case when G and H are connected and have the same number of nodes. However, we find a sufficient condition under which $G \times L_n \subseteq H \times L_n$ implies $G \subseteq H$.

Keywords: Cartesian product; Subgraph; Isomorphism; Embedding; Product graphs; Power graphs; Interconnection networks

1. Introduction

The Cartesian product has been found a useful tool to build large graphs from small factor graphs. For instance, there has been a number of interconnection networks proposed for parallel architectures that are, in fact, the Cartesian product of factor networks (e.g., [1,3,9]). Part of the interest of this class of networks is that many of their properties can be derived from the properties of the factor networks [2,11].

A very important property of an interconnection network is its capability of emulating other networks via embeddings. It is well known that the embedding properties of the factor networks propagate to the product network [6,p. 401,11]. For instance, if G can be efficiently embedded into H, then G^r (the r th Cartesian power of G) can be embedded into H^r with the same efficiency. However, to our knowledge, it is not known whether embedding properties of the product network imply similar embedding properties for the factor networks.

In this work, we start looking at this open question by considering containment between graphs, which is the simplest kind of embedding. Hence, the question we try to answer here is the following: "given that one product graph is a

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subgraph of another product graph, what can we say about their respective factor graphs?" In the particular case of product interconnection networks, answers to this question would allow to know whether two networks can be subgraph one of the other (and hence efficiently emulate one with the other) by only looking at their respective factor graphs.

1.1. Our results

We first look at power graphs, and show that, if G^r and H^r are isomorphic, then G and H are also isomorphic. This result could drive to conjecture that, if G^r is a subgraph of H^r, then G must be a subgraph of H. However, we disprove this conjecture by presenting counterexamples, even for the special cases when G and H are prime, and when they are connected and have the same number of nodes.

We then present a sufficient condition under which the containment of product graphs implies the containment of the factor graphs. We show that, if the product of *n connected* graphs $G_1, \ldots, G_n$ is a subgraph of the product of n graphs $H_1, \ldots, H_n$ *without* 4*-cycles*, then each graph G_i is a subgraph of a different graph H_j. As a consequence, applying this result to power graphs, if G is connected and H has no 4-cycles, then $G^r \subseteq H^r$ implies $G \subseteq H$. Since a number of graphs used as factors to construct interconnection networks have no 4-cycles (except in specific instances), e.g., the linear array, the ring, any tree, the cube-connected cycles, the mesh of trees [6], or the Petersen graph [9], these results can be directly applied to products and powers of these graphs.

Finally, we focus on the study of products with the linear array. We find a sufficient condition under which the product of one graph G with the linear array L_n being a subgraph of the product of another graph H with the same linear array L_n implies $G \subseteq H$. However, $G \times L_n \subseteq H \times L_n$ does not imply $G \subseteq H$ in general, since we find a counterexample even for the special case when G and H are connected and have the same number of nodes.

2. Definitions

All the graphs considered in this paper are finite undirected graphs without loops. We usually denote a graph by a capital letter, e.g., G. The set of vertices of a graph G is denoted as V_G and the set of edges as E_G. For simplicity, we denote the number of nodes of a graph G by $|G|$.

Although for convenience we make extensive use of labeled graphs, when comparing for containment or equality we consider all the graphs unlabeled, and therefore, we identify isomorphic graphs.

We start by formally defining the *Cartesian product* of two graphs.

Definition 1. The *Cartesian product* of two factor graphs $G=(V_G, E_G)$ and $H=(V_H, E_H)$ is the graph $G \times H$ whose vertex set is $V_G \times V_H$ and whose edge set contains all the edges $(uv, u'v')$ such that $\{u, u'\} \subseteq V_G$, $\{v, v'\} \subseteq V_H$, and either $u=u'$ and $(v, v') \in E_H$, or $v=v'$ and $(u, u') \in E_G$.

In the following sections, we will frequently abbreviate the Cartesian product of two graphs G and H ($G \times H$) as GH. This definition is extended to the product of more than two graphs in the obvious way. From Definition 1 it is easy to see that the Cartesian product operation is commutative (since we identify isomorphic graphs) and associative.

It can be observed that, given a fixed node $v \in V_H$, all the nodes $uv \in V_{G\times H}$ and the edges connecting them form a subgraph of $G \times H$ isomorphic to G. Clearly, there are $|H|$ disjoint such subgraphs, each uniquely identified by one node v of H. We say, then, that $G \times H$ contains $|H|$ disjoint copies of G and we denote the copy identified with the node $v \in V_H$ as Gv. The set of all the edges in the copies of G is denoted as the G-edges or the G-dimension. Similarly, $G \times H$ contains $|G|$ disjoint copies of H. We use a similar notation to identify each of them and to refer to the H-edges and H-dimension.

Now, we define the *direct sum* (also known as *union* of two graphs).

Definition 2. The *direct sum* of two component graphs $G=(V_G, E_G)$ and $H=(V_H, E_H)$ is the graph $G+H$ whose vertex set and edge set are the disjoint unions, respectively, of the vertex sets and edge sets of G and H.

If G has m connected components and H has n connected components, then $G+H$ will have $m+n$ connected components: m isomorphic to those of G and n isomorphic to those of H.

It is easy to see that the direct sum is commutative and associative. Furthermore, the Cartesian product is distributive with respect to the direct sum. In order to abbreviate expressions we will denote the graph with n components, all of them isomorphic to G, as nG.

We will also define some special classes of graphs that will be used in the rest of the paper.

Definition 3. The *trivial graph* is the graph $T = (V_T, E_T)$ such that V_T has exactly one vertex and E_T is the empty set.

Definition 4. The *null graph* is the graph $N = (V_N, E_N)$ such that V_N and E_N are the empty set.

Clearly, $G + N = G$, $G \times N = N$ and $G \times T = G$ for any graph G.

Definition 5. The *n-node linear array*, denoted L_n, is the graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{(i, i+1) : i \in \{0, \ldots, n-2\}\}$.

Definition 6. The *n-node ring*, denoted R_n, is the graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{(i, (i+1) \bmod n) : i \in \{0, \ldots, n-1\}\}$.

Definition 7 (*Leighton [6], p. 21*). The *bisection width* of a graph is the minimum number of edges which must be removed in order to split the graph into two disconnected subgraphs of equal (within one) number of nodes.

We denote as $B(G, a)$ the minimum number of edges that have to be removed from a graph G to disconnect it into two subgraphs G_1 and G_2 such that $||G_1| - |G_2|| \leqslant a$. Clearly, $B(G, a)$ is never larger than the bisection width of G.

Finally, we define the *maximal congestion*, a property of connected graphs. To do so, we first introduce the concept of embedding of graphs. An *embedding* of a guest graph into a host graph is a one-to-one mapping of the vertices of the guest into the vertices of the host and a mapping of the edges of the guest into paths of the host connecting the corresponding vertices. The *congestion* of an embedding is the maximum number of such paths that traverse any edge of the host.

Definition 8. The *maximal congestion* of a connected graph G, denoted as $C(G)$, is the minimum congestion of an embedding of the $|G|$-node directed complete graph (i.e., one in which each pair of nodes is connected by two arcs with opposite orientations) onto G.

3. Containment of power graphs

This section is devoted to study containment results among power graphs. First, we show that if two power graphs with the same number of dimensions are isomorphic, then their respective factor graphs must be isomorphic as well. We continue by showing that this does not hold for containment, i.e., the fact of a power graph being a subgraph of another power graph with the same number of dimensions does not imply that their respective factor graphs are contained one in the other.

Sabidussi has shown in [10] that a connected graph has a unique factorization into a multi-set of Cartesian-prime graphs. From this property, it is easy to see that if G and H are two connected graphs, $G^r = H^r$ implies $G = H$. However, this may be not so straightforward for disconnected graphs since they do not have a unique factorization, as it is shown below.

Certificates (canonical forms) of graphs have been widely used for graph isomorphism testing. A certificate is a numeric value such that two graphs have the same certificate if and only if they are isomorphic. The technique used for isomorphism testing first computes the certificates of the graphs to be tested, and then compares these certificates for equality. For more information on computing certificates, see, for example, [4, Chapter 7]. Then, we can use certificates to totally order all non-trivial connected Cartesian-prime graphs, and use this ordering to enumerate all these graphs as $F_1, F_2, \ldots$.

Let $X = \{x_1, x_2, \ldots\}$ be a denumerable set of (commutative) variables, and let $\mathscr{R} = \mathbb{Z}[X]$ be the integral polynomial ring in these variables. We can define a correspondence between F_i and x_i, $i \geqslant 1$. Let the trivial graph T correspond to the trivial monomial $1 \in \mathscr{R}$ and the null graph N to $0 \in \mathscr{R}$. Then, each graph corresponds to a polynomial in $\mathscr{R}$ with non-negative coefficients, and vice-versa. We denote by $P(G)$ the polynomial associated to a graph G, and $P^{-1}(p)$ the graph associated to a polynomial p in $\mathscr{R}$ with non-negative coefficients. Note that the Cartesian product of graphs corresponds to the multiplication in $\mathscr{R}$ and the direct sum of graphs to the polynomial sum. Clearly, for graphs G and H we have that $P(G \times H) = P(G)P(H)$ and $P(G + H) = P(G) + P(H)$.

We first observe that, unlike $\mathscr{R}$, the set of polynomials with non-negative coefficients in $\mathscr{R}$ is not a unique factorization domain, as shown by Nakayama and Hashimoto [7]. See, for instance, the following simple example due to Nüsken [8]. The polynomial $p(x_1) = x_1^4 + 2x_1^3 + x_1^2 + 4x_1 + 4$ on $x_1 \in X$ is a polynomial in $\mathscr{R}$ with non-negative coefficients. Note that the unique factorization of $p(x_1)$ in $\mathscr{R}$ yields $x_1^4 + 2x_1^3 + x_1^2 + 4x_1 + 4 = (x_1 + 1)(x_1^2 - x_1 + 2)(x_1 + 2)$, which has a factor with a negative coefficient. Without negative coefficients we obtain two different factorizations $(x_1^3 + x_1 + 2)(x_1 + 2) = (x_1 + 1)(x_1^3 + x_1^2 + 4)$. From this fact, it follows that the graph $P^{-1}(p(x_1))$ has two different prime factorizations.

Theorem 1. *$G^r = H^r$ implies $G = H$.*

Proof. Let S be the multi-set of prime polynomials obtained from the *unique* factorization of $P(G)$ in $\mathscr{R}$. Note that not all the polynomials in S must have non-negative coefficients, and hence correspond to a graph. Since $P(G^r) = (P(G))^r$, the multi-set $U = \bigcup_{i=1}^r S$ satisfies that $\prod_{u \in U} u = P(G^r)$. Furthermore, U is a factorization of $P(G^r)$ in $\mathscr{R}$, since all polynomials in U are prime. Finally, U is the *unique* factorization of $P(G^r)$ in $\mathscr{R}$, since $\mathscr{R}$ is a unique factorization domain.

Similarly, let T be the multi-set of prime polynomials obtained from the *unique* factorization of $P(H)$ in $\mathscr{R}$. Then, $\bigcup_{i=1}^r T$ is the unique factorization of $P(H^r)$ in $\mathscr{R}$. Since $P(G^r) = P(H^r)$, we must have that $U = \bigcup_{i=1}^r S = \bigcup_{i=1}^r T$, and hence $S = T$. Therefore, $P(G) = P(H)$ and $G = H$. □

This theorem could lead us to conjecture that a similar result holds when G^r and H^r are not isomorphic, but subgraphs one of the other. The following theorem disproves this conjecture.

Theorem 2. *$G^r \subseteq H^r$ does not imply $G \subseteq H$.*

Proof. To prove the theorem we find two graphs G and H such that G is not a subgraph of H but G^2 is a subgraph of H^2. Let us consider the graphs K, I, and J, presented in Fig. 1:

$$K = L_4 \times R_3, \quad I = R_3^2, \quad J = L_4^2.$$

Clearly, K is not a subgraph of J, since J does not contain R_3 as a subgraph. Similarly, K is not a subgraph of I since K has 12 nodes while I only has 9.

However, observe that $K^2 = L_4^2 \times R_3^2 = J \times I$. Then, if we define $G = K$ and $H = J + I$, we have two graphs G and H such that G^2 is contained in H^2 but G is not a subgraph of H. It is easy to see that $G^2 = IJ \subseteq I^2 + 2IJ + J^2 = H^2$. □

Note that the counterexample we just presented uses product and disconnected graphs. Hence, it does not cover the case when G and H are both prime or when they are connected. Counterexamples for these special cases can be obtained by slightly modifying the one above.

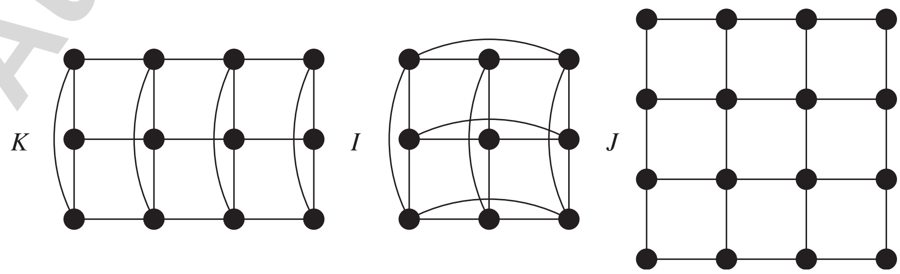

Fig. 1. Graphs *K*, *I*, and *J*.

Corollary 1. *Let G and H be connected graphs, even then* $G^r \subseteq H^r$ *does not imply* $G \subseteq H$.

Proof. Let K, J, I, and G be as defined in the previous proof. Note that G is already connected. We can construct a connected graph H by connecting J to I using just one edge (any will do).

G is not a subgraph of the graph H constructed, since it is neither a subgraph of J nor I, and the minimum cut in G contains three edges. However, G^2 is a subgraph of H^2 since this H is a super-graph of the H in the previous proof. □

Corollary 2. *Let G and H be connected prime graphs, even then* $G^r \subseteq H^r$ *does not imply* $G \subseteq H$.

Proof. The graph G in the previous proof can be made prime by deleting any edge from it. The graph H from the previous proof is already prime because it has a bridge. □

The counterexamples just presented do not cover a large special class of graphs: graphs with the same number of nodes. A counterexample for this special case can be again obtained by adapting the previous ones.

Corollary 3. *Let* $|G| = |H|$, *even then* $G^r \subseteq H^r$ *does not imply* $G \subseteq H$.

Proof. Let K, J, and I be as defined in the proof of Theorem 2. We will construct two graphs G and H with the same number of nodes such that G is not a subgraph of H but G^2 is contained in H^2.

We first make $H = 2I + J$. Trivially, H has exactly 34 nodes. Then, we make $G = K + 22T$ (one copy of K and exactly 22 isolated nodes). Clearly, G and H have the same number of nodes and G is not contained in H. Clearly, $G^2 = K^2 + 44K + 22^2T$ and $H^2 = 4I^2 + 4IJ + J^2$. Since $K^2 = IJ$, then $12K \subseteq IJ$ (remember that from the construction of the Cartesian product, K^2 has $|K| = 12$ disjoint copies of K). Additionally, $6K \subseteq I^2$ as shown in Fig. 2. Therefore, $K^2 \subseteq IJ$, $36K \subseteq 3IJ$ and $8K \subseteq 2I^2$. This makes $K^2 + 44K \subseteq 4IJ + 2I^2$. Finally, the 22^2 isolated nodes in G^2 can then be mapped to the rest of the nodes in H^2. This shows that G^2 is a subgraph of H^2 and completes the proof. □

Finally, trying to restrict even more the class of graphs in order to find a positive result, we choose the class of connected graphs with the same number of nodes. However, even for this very restrictive case we have found a counterexample, shown in the next theorem.

Theorem 3. *Let G and H be connected graphs such that* $|G| = |H|$, *even then* $G^r \subseteq H^r$ *does not imply* $G \subseteq H$.

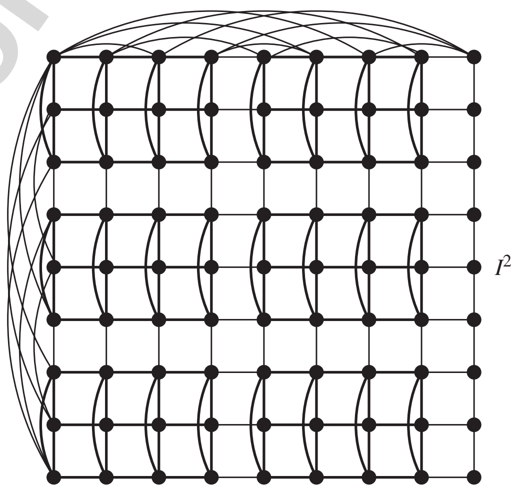

Fig. 2. Graph I^2 contains six disjoint copies of K. For the sake of clarity, some edges have been removed from all but the top row and all but the leftmost column.

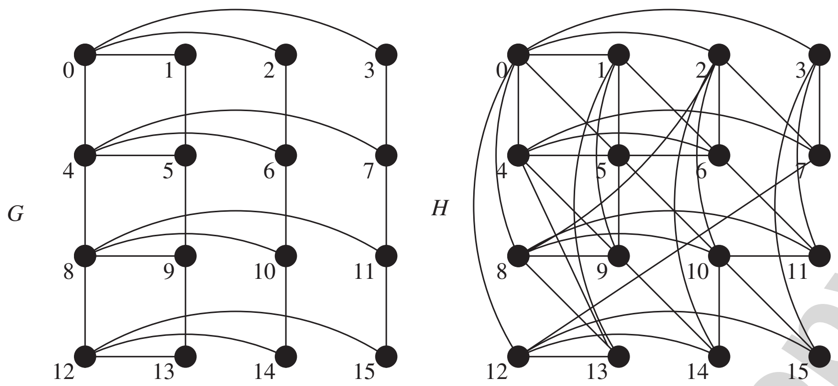

Fig. 3. Graphs G and H of Theorem 3.

Proof. Let us consider the graphs G and H as shown in Fig. 3. By exhaustive search and with the aid of a computer program we have found that while $G \nsubseteq H$, $G^2 \subseteq H^2$. The mapping of the vertices of G^2 into the vertices of H^2 is presented in the Appendix. The input files with graphs G and H, along with the programs that have been used to compute their respective squares, to find the mapping, and to verify it are available at URL http://www.diatel.upm.es/jllopez/containment. □

4. Containment of products of graphs

In this section, we give a sufficient condition for a collection of factor graphs $G_1, \ldots, G_n$ to be pairwise subgraph of another collection $H_1, \ldots, H_n$ when their respective products are. To simplify the notation, we use $\mathscr{G}$ to denote $\prod_{i=1}^{n} G_i$ and $\mathscr{H}$ to denote $\prod_{i=1}^{n} H_i$, respectively. In most of the results presented we assume that $\mathscr{G} \subseteq \mathscr{H}$ under some mapping. We will use $\mathscr{G} \subseteq_M \mathscr{H}$ to explicitly expose the mapping M that satisfies the containment property, when needed.

In order to prove the main result of this section, we need some lemmas.

Lemma 1. *Let $G_1, \ldots, G_n$ be a collection of n connected graphs and let $H_1, \ldots, H_n$ be another collection of n graphs. If $\mathscr{G} \subseteq_M \mathscr{H}$, then all the edges connecting any two adjacent copies of G_i, $i \in \{1, \ldots, n\}$, in $\mathscr{G}$ are mapped under M to the same dimension in $\mathscr{H}$.*

Proof. Let (a_i, b_i) denote an edge of the graph G_i, for any $i \in \{1, \ldots, n\}$, and (u_j, v_j) and (u_j, w_j) denote adjacent edges of the graph H_j, for any $j \in \{1, \ldots, n\}$. Let us consider a 4-cycle in $\mathscr{G}$ formed by the G_1-edges $e = (a_1a_2a_3 \ldots a_n, b_1a_2a_3 \ldots a_n)$ and $e' = (a_1b_2a_3 \ldots a_n, b_1b_2a_3 \ldots a_n)$, and the G_2-edges $\alpha = (a_1a_2a_3 \ldots a_n, a_1b_2a_3 \ldots a_n)$ and $\alpha' = (b_1a_2a_3 \ldots a_n, b_1b_2a_3 \ldots a_n)$. Without loss of generality, we assume that e is mapped to the H_1-edge $M(e) = (u_1u_2u_3 \ldots u_n, v_1u_2u_3 \ldots u_n)$. By the way of contradiction, let us assume that α and α' are mapped to edges in different dimensions in $\mathscr{H}$. We need to consider two cases.

Case 1: One of the G_2-edges is mapped to an H_1-edge and the other is mapped to some other dimension. W.l.o.g., we can assume that α is mapped to the H_1-edge $M(\alpha) = (u_1u_2u_3 \ldots u_n, w_1u_2u_3 \ldots u_n)$ and α' is mapped to the H_2-edge $M(\alpha') = (v_1u_2u_3 \ldots u_n, v_1v_2u_3 \ldots u_n)$. Then, e' should be mapped to the edge $M(e') = (w_1u_2u_3 \ldots u_n, v_1v_2u_3 \ldots u_n)$, which does not exist in $\mathscr{H}$ from the construction of the Cartesian product, and we reach a contradiction.

Case 2: Both G_2-edges are mapped to dimensions different from H_1. W.l.o.g., we can assume that α is mapped to the H_2-edge $M(\alpha) = (u_1u_2u_3 \ldots u_n, u_1v_2u_3 \ldots u_n)$ and α' is mapped to the H_3-edge $M(\alpha') = (v_1u_2u_3 \ldots u_n, v_1u_2v_3 \ldots u_n)$. Then, e' should be mapped to the edge $M(e') = (u_1v_2u_3 \ldots u_n, v_1u_2v_3 \ldots u_n)$, which does not exist in $\mathscr{H}$ from the construction of the Cartesian product, and we reach a contradiction.

Therefore, both edges α and α' must be mapped to edges in the same dimension in $\mathscr{H}$. Since all the graphs $G_1, \ldots, G_n$ are connected, the above argument can be propagated to show that all the edges connecting two copies of G_i in $\mathscr{G}$ must be mapped to edges in the same dimension in $\mathscr{H}$. □

Lemma 2. *Let $H_1, \ldots, H_n$ be a collection of n graphs without* 4*-cycles. Then, every* 4*-cycle in $\mathscr{H}$ is built from two non-incident edges in one dimension H_j and two non-incident edges in another dimension H_k, where $j, k \in \{1, \ldots, n\}$ and $j \neq k$.*

Proof. Let (a_i, b_i), (a_i, c_i), and (c_i, d_i) be possible edges in any graph H_i, $i \in \{1, \ldots, n\}$. Let us assume, without loss of generality, that the H_1 edge $(a_1a_2 \ldots a_n, b_1a_2 \ldots a_n)$ belongs to a 4-cycle in $\mathscr{H}$. We will follow the other possible edges that form that 4-cycle, starting from node $a_1a_2 \ldots a_n$. We consider two possibilities for the next edge in the cycle incident to node $b_1a_2 \ldots a_n$.

Case 1: The next edge is an edge $(b_1a_2 \ldots a_n, c_1a_2 \ldots a_n)$ in the same dimension H_1. Then, we have two possibilities for the next edge in the cycle.

Case 1.1: The next edge is an edge $(c_1a_2 \ldots a_n, d_1a_2 \ldots a_n)$ in the same dimension H_1. In this case, to complete a 4-cycle the fourth edge must be the edge $(d_1a_2 \ldots a_n, a_1a_2 \ldots a_n)$. However, this is not possible since H_1 does not contain 4-cycles by definition.

Case 1.2: The next edge is an edge in some other dimension (say H_2, without loss of generality). Let the edge be $(c_1a_2a_3 \ldots a_n, c_1b_2a_3 \ldots a_n)$. In this case, to complete a 4-cycle the fourth edge must be the edge $(c_1b_2a_3 \ldots a_n, a_1a_2a_3 \ldots a_n)$. However, this is not possible since this is not an edge in $\mathscr{H}$ from the construction of the Cartesian product.

Case 2: The next edge is an edge in some dimension different from H_1 (say H_2, without loss of generality). Let the edge be $(b_1a_2a_3 \ldots a_n, b_1b_2a_3 \ldots a_n)$. Then, we have four possibilities for the next edge in the cycle.

Case 2.1: The next edge is an edge $(b_1b_2a_3 \ldots a_n, a_1b_2a_3 \ldots a_n)$ in the dimension H_1. In such a case, the fourth edge should be the edge $(a_1b_2a_3 \ldots a_n, a_1a_2a_3 \ldots a_n)$, which is in fact an edge in $\mathscr{H}$.

Case 2.2: The next edge is an edge $(b_1b_2a_3 \ldots a_n, c_1b_2a_3 \ldots a_n)$ in the dimension H_1. In such a case, the fourth edge should be the edge $(c_1b_2a_3 \ldots a_n, a_1a_2a_3 \ldots a_n)$. However, this is not possible since this is not an edge in $\mathscr{H}$ from the construction of the Cartesian product.

Case 2.3: The next edge is an edge in the dimension H_2. This is impossible by Case 1.

Case 2.4: The next edge is an edge in a different dimension from H_1 and H_2 (say H_3, without loss of generality). Let the edge be $(b_1b_2a_3a_4 \ldots a_n, b_1b_2b_3a_4 \ldots a_n)$. In such a case, the fourth edge should be the edge $(b_1b_2b_3a_4 \ldots a_n, a_1a_2a_3a_4 \ldots a_n)$. However, this is not possible since this is not an edge in $\mathscr{H}$ from the construction of the Cartesian product.

Therefore, any 4-cycle in $\mathscr{H}$ has two non-adjacent edges in one dimension and the other two in a different dimension. □

Lemma 3. *Let $G_1, \ldots, G_n$ be a collection of n connected graphs, let $H_1, \ldots, H_n$ be a collection of n graphs without 4-cycles, and let e be an edge of G_i, $i \in \{1, \ldots, n\}$. If $\mathscr{G} \subseteq_M \mathscr{H}$ then, under mapping M, all the e edges in all copies of G_i in $\mathscr{G}$ are mapped to the same dimension in $\mathscr{H}$.*

Proof. From Lemma 1, the edges e of two adjacent copies of G_i in $\mathscr{G}$ are mapped into $\mathscr{H}$ in such a way that they are connected by edges in the same dimension forming 4-cycles. From Lemma 2, these 4-cycles have two non-adjacent edges in one dimension and the other two in a different dimension. Then, when one copy of G_i has e mapped into one dimension, in any adjacent copy e must be mapped to the same dimension. Since the graphs $G_1, \ldots, G_n$ are connected, the above argument may be extended to every copy of G_i. □

Theorem 4. *Let $G_1, \ldots, G_n$ be n connected graphs with at least one edge and let $H_1, \ldots, H_n$ be n graphs without 4-cycles. Then, $\mathscr{G} \subseteq_M \mathscr{H}$ implies that there exists a one-to-one correspondence f between $\{G_1, \ldots, G_n\}$ and $\{H_1, \ldots, H_n\}$ such that $G_i \subseteq f(G_i)$, $i \in \{1, \ldots, n\}$.*

Proof. Let $\mathscr{G} \subseteq \mathscr{H}$ as defined by mapping M and assume, by the way of contradiction, that either f is not defined for some $G_i \in \{G_1, \ldots, G_n\}$ or f is fully defined but it is not a one-to-one correspondence between $\{G_1, \ldots, G_n\}$ and $\{H_1, \ldots, H_n\}$. We will consider these two cases separately.

Case 1: f is not defined for some $G_i \in \{G_1, \ldots, G_n\}$, i.e., there is a G_i that is not a subgraph of any H_j. Let us assume, without loss of generality, that no copy of G_1 is mapped, under mapping M, completely into a copy of some graph H_j. Note that, from Lemma 3, all the copies of a given edge e of G_1 are mapped into edges of the same dimension in $\mathscr{H}$, i.e., they are all mapped to H_j-edges, for some $j \in \{1, \ldots, n\}$. Hence, there are two adjacent edges of G_1 mapped to two different dimensions in $\mathscr{H}$, and these two dimensions are the same for all the copies of G_1. Let us assume, without loss of generality, that these dimensions are H_1 and H_2. We will denote the subgraph of G_1 formed by these two edges and their attached nodes by G_1'.

We will take now a subgraph of $\mathscr{G}$, which we will denote $\mathscr{G}'$, by taking just a subgraph G'_i with two adjacent nodes, and the edge connecting them, of every graph G_i, $i \in \{2, \ldots, n\}$, and defining $\mathscr{G}' = \prod_{i=1}^{n} G'_i$. Clearly, $\mathscr{G}' \subseteq \mathscr{G}$, and the mapping M is still defined for the nodes of $\mathscr{G}'$ and implies that $\mathscr{G}' \subseteq \mathscr{H}$.

Observe now that in $\mathscr{G}'$ a given copy of G'_1 is connected to $n-1$ other copies of G'_1 by means of G'_i-edges, $i \in \{2, \ldots, n\}$. Also, the G'_i-edges with those of G'_1 form 4-cycles, from the construction of the Cartesian product. From Lemma 1, all the edges connecting two copies of G'_1 must be mapped to edges in the same dimension in $\mathscr{H}$ and, from Lemma 2, all the 4-cycles in $\mathscr{H}$ are formed by two edges in one dimension (facing each other) and two in a different dimension. Hence, since the edges of all the copies of G'_1 are mapped to edges of H_1 and H_2, the G'_i-edges can only be mapped to dimensions $H_3, \ldots, H_n$. That means that the edges of at least two dimensions G'_i and G'_j, where $i, j \in \{2, \ldots, n\}$ and $i \neq j$, are mapped to the same dimension H_l, $l \in \{3, \ldots, n\}$.

However, from the construction of the Cartesian product, any two adjacent edges of G'_i and G'_j are part of a 4-cycle in $\mathscr{G}'$, and from Lemma 2 have to be mapped to different dimensions in $\mathscr{H}$. Thus, we have come to a contradiction and we can conclude that f must be defined for every $G_i \in \{G_1, \ldots, G_n\}$.

Case 2: Assume now that f is defined for every $G_i \in \{G_1, \ldots, G_n\}$ but it is not a one-to-one correspondence, that is, for some $i, j \in \{1, \ldots, n\}$, $i \neq j$, $f(G_i) = f(G_j)$. This means that, under mapping M, all the edges of all the copies of two different graphs G_i and G_j have been mapped to the same dimension H_l in $\mathscr{H}$, where $i, j, l \in \{1, \ldots, n\}$. Since the copies of H_l in $\mathscr{H}$ are disjoint, we must have that $G_i \times G_j \subseteq H_l$. Since G_i and G_j have at least one edge, $G_i \times G_j$ must have at least one 4-cycle. Therefore, H_l must also have at least one 4-cycle and we reach a contradiction. Hence, f must be a one-to-one correspondence. □

Corollary 4. *Let G be a connected graph and H a graph without* 4-*cycles, then $G^r \subseteq H^r$ implies $G \subseteq H$.*

Note that a number of graphs used as factors to construct interconnection networks have no 4-cycles (except in specific instances), e.g., the linear array, the ring, any tree, the cube-connected cycles, the mesh of trees [6], or the Petersen graph [9]. Hence, these results can be applied to products or powers of these graphs.

5. Containment on products with the linear array

In this section, we explore containment results for products of arbitrary graphs G and H with the same linear array L_n. We first present one condition that makes G be contained in H if $G \times L_n$ is contained in $H \times L_n$. Then, we end the section by showing that, in general, $G \times L_n \subseteq H \times L_n$ does not imply $G \subseteq H$.

The following theorem presents a condition that guarantees the containment of the factor graphs. Recall that $B(G, a)$ is the minimum number of edges that have to be removed to break G into two subgraphs that differ in at most a nodes.

Theorem 5. *Let G be connected and $|H| < n \cdot B(G, \lfloor |H|/n \rfloor)$, then $G \times L_n \subseteq H \times L_n$ implies $G \subseteq H$.*

Proof. We start by showing that if after mapping $G \times L_n$ into $H \times L_n$ two adjacent copies of G are not connected by H-edges, then $G \subseteq H$. Then, we derive that, if that is not the case, each copy of H contains the same subset of nodes from all the copies of G, leading to the result.

First, assume that some L_n-edge of $G \times L_n$ is mapped to an L_n-edge in $H \times L_n$. Without loss of generality, we can assume that it is edge (u_1, u_2), where $u \in V_G$. Then, from Lemma 1, all the edges connecting the G-copies G_1 and G_2 are mapped to L_n-edges.

Furthermore, all the edges of these copies of G have to be mapped to H-edges. This can be easily shown by contradiction. Let us assume, without loss of generality, that (u_1, u_2) is mapped to the edge (x_1, x_2), $x \in V_H$, and that $(u, v) \in E_G$. Let us assume (u_1, v_1) is not mapped to an H-edge. Then, it has to be mapped to the edge (x_0, x_1), because node u_2 is already mapped to x_2. But, independently of how the edge (u_2, v_2) is mapped, it is not possible that the image of v_2 is connected to x_0, image of v_1. Therefore, we reach a contradiction and all the G-edges for the two copies are H-edges. This implies that H contains all the edges of G, and $G \subseteq H$.

Let us assume now that all the L_n-edges of $G \times L_n$ are mapped to H-edges in $H \times L_n$. Therefore, each copy of L_n in $G \times L_n$ is fully mapped into a copy of H. Clearly, the maximum number of copies of L_n that can be contained in one copy of H is $\lfloor |H|/n \rfloor$. We will denote this value as a for brevity.

Thus, each copy of H contains the same subset of nodes from all the copies of G. The maximum number of nodes in the subset is a. We can define, then, n disjoint subgraphs of G, namely $G_0, \ldots, G_{n-1}$, where G_i is the subgraph of each copy of G mapped to the copy H_i. Since H_i is only connected to $H(i-1)$ and $H(i+1)$, G_i is only connected to G_{i-1} and G_{i+1} in G.

In any case, the number of edges connecting two adjacent copies of H is $|H|$. Now, since each copy of H hosts up to a nodes from each copy of G, it is possible to find an index p such that $\sum_{i=0}^{p-1} |G_i| - \sum_{i=p}^{n-1} |G_i| \leqslant a$. The number of edges connecting G_{p-1} and G_p is at least $B(G, a)$.

Thus, since each H_i holds n copies of G_i, there must be at least $n\ B(G, a)$ edges connecting $H(p-1)$ and Hp. Therefore, $|H| \geqslant n\ B(G, a)$. □

Since the exact value of $B(G, \lfloor |H|/n \rfloor)$ may not be easy to obtain, we can use the maximal congestion $C(G)$ to obtain a lower bound on it. This yields the following corollary.

Corollary 5. *Let G be connected and $|H| < n \cdot |G|^2 - \lfloor \frac{|H|}{n} \rfloor^2 / 2C(G)$, then $G \times L_n \subseteq H \times L_n$ implies $G \subseteq H$.*

Proof. We simply obtain a lower bound on $B(G, a)$ by using the maximal congestion $C(G)$. The result follows from this bound and Theorem 5.

We know that the number of edges connecting a subgraph with k nodes of the $|G|$-node directed complete graph with the rest of the graph is $2k(|G| - k)$. Given the definition of maximal congestion, it is easy to see that the number of edges in G connecting a subgraph with k nodes with the rest of the graph is at least $2k(|G| - k)/C(G)$. Without loss of generality we can assume $k \leqslant |G| - k$.

Note that the closer the values k and $|G| - k$ are, the larger this bound is. To obtain a lower bound we can assume their difference is exactly a, the largest allowed. Then $k = (|G| - a)/2$ and $|G| - k = (|G| + a)/2$. This yields the lower bound $B(G, a) \geqslant (|G|^2 - a^2)/2C(G)$. □

Upper bounds on the maximal congestion for many popular graphs are easily obtained [6,5]. Those bounds can be used with the above corollary. For other graphs different lower bounds on $B(G, a)$ can be used, like the minimum cut of G.

Given this result we could expect that maybe $G \times L_n \subseteq H \times L_n$ always implies $G \subseteq H$. The following result shows that this is not the case, even for special classes of graphs.

Theorem 6. *Let G and H be connected graphs and $|G| = |H|$, even then $G \times L_2 \subseteq H \times L_2$ does not imply $G \subseteq H$.*

Proof. We prove the claim by presenting two graphs G and H such that G is not a subgraph of H but $G \times L_2$ is a subgraph of $H \times L_2$. Fig. 4 presents such graphs.

By inspection it can be seen that G is not contained in H. G has two 4-cycles with one diagonal, connected with one middle node from nodes incident to the diagonal. It can be seen that in H there are only four 4-cycles with diagonals, and that those connected with a middle node share one node. Therefore, G is not in H.

However, $G \times L_2$ is contained in $H \times L_2$ as shown in Fig. 5. This figure shows the two copies of H contained in $H \times L_2$ and only the necessary edges connecting them. The graph $G \times L_2$ contained in the subgraph shown has been highlighted, presenting the G-edges with thick solid lines, and the L_n-edges with dashed lines.

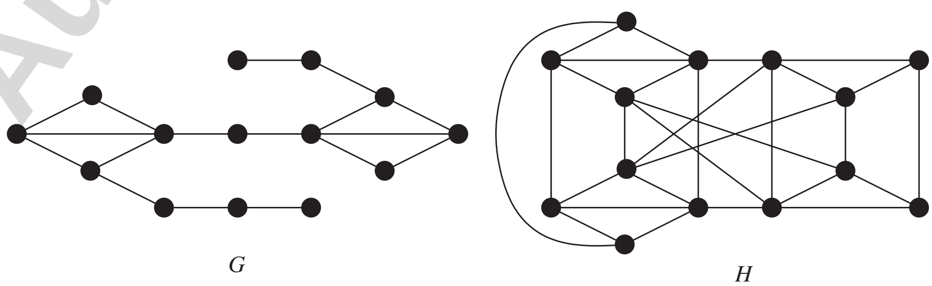

Fig. 4. Graphs G and H.

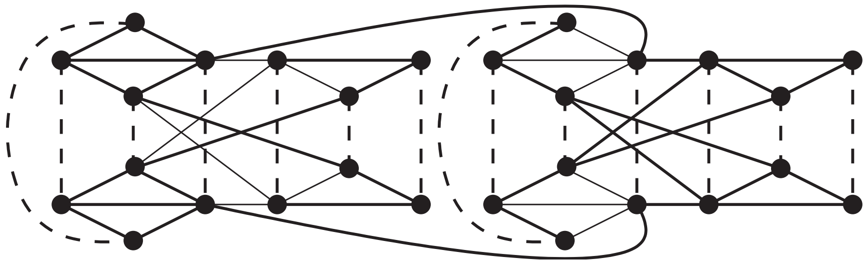

Fig. 5. $G \times L_2$ is contained in $H \times L_2$. Some edges from $H \times L_2$ have been omitted for the sake of clarity.

Therefore, G is not contained in H but $G \times L_2$ is contained in $H \times L_2$, and the proof is complete. □

6. Conclusions

In this paper, we study the containment properties of factor graphs given the containment of product and power graphs, presenting positive and negative results. We show here that it is possible in some cases to derive containment properties of the factors given the containment of the products.

There are several interesting questions that remain open after this work. For instance, it would be nice to find simpler sufficient conditions for containment than the ones described in Theorem 4 and Corollary 4. Another interesting line of work is to find embedding properties for the factor graphs derived from the embedding properties of the products (i.e., to relax the unit dilation and congestion requirement of containment).

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Appendix A. Mapping for the proof of Theorem 3

The following is a mapping of the vertices of graph G^2 into those of graph H^2 found to prove Theorem 3.

$(0,0) \to (0,1)$	$(4,0) \to (0,6)$	$(8,0) \to (0,11)$	$(12,0) \to (0,3)$
$(0,1) \to (1,1)$	$(4,1) \to (1,6)$	$(8,1) \to (1,11)$	$(12,1) \to (1,3)$
$(0,2) \to (2,1)$	$(4,2) \to (2,6)$	$(8,2) \to (2,11)$	$(12,2) \to (2,3)$
$(0,3) \to (3,1)$	$(4,3) \to (3,6)$	$(8,3) \to (3,11)$	$(12,3) \to (3,3)$
$(0,4) \to (0,0)$	$(4,4) \to (0,5)$	$(8,4) \to (0,10)$	$(12,4) \to (0,15)$
$(0,5) \to (1,0)$	$(4,5) \to (1,5)$	$(8,5) \to (1,10)$	$(12,5) \to (1,15)$
$(0,6) \to (2,0)$	$(4,6) \to (2,5)$	$(8,6) \to (2,10)$	$(12,6) \to (2,15)$
$(0,7) \to (3,0)$	$(4,7) \to (3,5)$	$(8,7) \to (3,10)$	$(12,7) \to (3,15)$
$(0,8) \to (0,4)$	$(4,8) \to (0,9)$	$(8,8) \to (0,14)$	$(12,8) \to (0,12)$
$(0,9) \to (1,4)$	$(4,9) \to (1,9)$	$(8,9) \to (1,14)$	$(12,9) \to (1,12)$
$(0,10) \to (2,4)$	$(4,10) \to (2,9)$	$(8,10) \to (2,14)$	$(12,10) \to (2,12)$
$(0,11) \to (3,4)$	$(4,11) \to (3,9)$	$(8,11) \to (3,14)$	$(12,11) \to (3,12)$
$(0,12) \to (0,13)$	$(4,12) \to (0,8)$	$(8,12) \to (0,2)$	$(12,12) \to (0,7)$
$(0,13) \to (1,13)$	$(4,13) \to (1,8)$	$(8,13) \to (1,2)$	$(12,13) \to (1,7)$
$(0,14) \to (2,13)$	$(4,14) \to (2,8)$	$(8,14) \to (2,2)$	$(12,14) \to (2,7)$
$(0,15) \to (3,13)$	$(4,15) \to (3,8)$	$(8,15) \to (3,2)$	$(12,15) \to (3,7)$
$(1,0) \to (4,1)$	$(5,0) \to (4,6)$	$(9,0) \to (4,11)$	$(13,0) \to (4,3)$
$(1,1) \to (5,1)$	$(5,1) \to (5,6)$	$(9,1) \to (5,11)$	$(13,1) \to (5,3)$
$(1,2) \to (6,1)$	$(5,2) \to (6,6)$	$(9,2) \to (6,11)$	$(13,2) \to (6,3)$

$(1, 3) \to (7, 1)$	$(5, 3) \to (7, 6)$	$(9, 3) \to (7, 11)$	$(13, 3) \to (7, 3)$
$(1, 4) \to (4, 0)$	$(5, 4) \to (4, 5)$	$(9, 4) \to (4, 10)$	$(13, 4) \to (4, 15)$
$(1, 5) \to (5, 0)$	$(5, 5) \to (5, 5)$	$(9, 5) \to (5, 10)$	$(13, 5) \to (5, 15)$
$(1, 6) \to (6, 0)$	$(5, 6) \to (6, 5)$	$(9, 6) \to (6, 10)$	$(13, 6) \to (6, 15)$
$(1, 7) \to (7, 0)$	$(5, 7) \to (7, 5)$	$(9, 7) \to (7, 10)$	$(13, 7) \to (7, 15)$
$(1, 8) \to (4, 4)$	$(5, 8) \to (4, 9)$	$(9, 8) \to (4, 14)$	$(13, 8) \to (4, 12)$
$(1, 9) \to (5, 4)$	$(5, 9) \to (5, 9)$	$(9, 9) \to (5, 14)$	$(13, 9) \to (5, 12)$
$(1, 10) \to (6, 4)$	$(5, 10) \to (6, 9)$	$(9, 10) \to (6, 14)$	$(13, 10) \to (6, 12)$
$(1, 11) \to (7, 4)$	$(5, 11) \to (7, 9)$	$(9, 11) \to (7, 14)$	$(13, 11) \to (7, 12)$
$(1, 12) \to (4, 13)$	$(5, 12) \to (4, 8)$	$(9, 12) \to (4, 2)$	$(13, 12) \to (4, 7)$
$(1, 13) \to (5, 13)$	$(5, 13) \to (5, 8)$	$(9, 13) \to (5, 2)$	$(13, 13) \to (5, 7)$
$(1, 14) \to (6, 13)$	$(5, 14) \to (6, 8)$	$(9, 14) \to (6, 2)$	$(13, 14) \to (6, 7)$
$(1, 15) \to (7, 13)$	$(5, 15) \to (7, 8)$	$(9, 15) \to (7, 2)$	$(13, 15) \to (7, 7)$
$(2, 0) \to (8, 1)$	$(6, 0) \to (8, 6)$	$(10, 0) \to (8, 11)$	$(14, 0) \to (8, 3)$
$(2, 1) \to (9, 1)$	$(6, 1) \to (9, 6)$	$(10, 1) \to (9, 11)$	$(14, 1) \to (9, 3)$
$(2, 2) \to (10, 1)$	$(6, 2) \to (10, 6)$	$(10, 2) \to (10, 11)$	$(14, 2) \to (10, 3)$
$(2, 3) \to (11, 1)$	$(6, 3) \to (11, 6)$	$(10, 3) \to (11, 11)$	$(14, 3) \to (11, 3)$
$(2, 4) \to (8, 0)$	$(6, 4) \to (8, 5)$	$(10, 4) \to (8, 10)$	$(14, 4) \to (8, 15)$
$(2, 5) \to (9, 0)$	$(6, 5) \to (9, 5)$	$(10, 5) \to (9, 10)$	$(14, 5) \to (9, 15)$
$(2, 6) \to (10, 0)$	$(6, 6) \to (10, 5)$	$(10, 6) \to (10, 10)$	$(14, 6) \to (10, 15)$
$(2, 7) \to (11, 0)$	$(6, 7) \to (11, 5)$	$(10, 7) \to (11, 10)$	$(14, 7) \to (11, 15)$
$(2, 8) \to (8, 4)$	$(6, 8) \to (8, 9)$	$(10, 8) \to (8, 14)$	$(14, 8) \to (8, 12)$
$(2, 9) \to (9, 4)$	$(6, 9) \to (9, 9)$	$(10, 9) \to (9, 14)$	$(14, 9) \to (9, 12)$
$(2, 10) \to (10, 4)$	$(6, 10) \to (10, 9)$	$(10, 10) \to (10, 14)$	$(14, 10) \to (10, 12)$
$(2, 11) \to (11, 4)$	$(6, 11) \to (11, 9)$	$(10, 11) \to (11, 14)$	$(14, 11) \to (11, 12)$
$(2, 12) \to (8, 13)$	$(6, 12) \to (8, 8)$	$(10, 12) \to (8, 2)$	$(14, 12) \to (8, 7)$
$(2, 13) \to (9, 13)$	$(6, 13) \to (9, 8)$	$(10, 13) \to (9, 2)$	$(14, 13) \to (9, 7)$
$(2, 14) \to (10, 13)$	$(6, 14) \to (10, 8)$	$(10, 14) \to (10, 2)$	$(14, 14) \to (10, 7)$
$(2, 15) \to (11, 13)$	$(6, 15) \to (11, 8)$	$(10, 15) \to (11, 2)$	$(14, 15) \to (11, 7)$
$(3, 0) \to (12, 1)$	$(7, 0) \to (12, 6)$	$(11, 0) \to (12, 11)$	$(15, 0) \to (12, 3)$
$(3, 1) \to (13, 1)$	$(7, 1) \to (13, 6)$	$(11, 1) \to (13, 11)$	$(15, 1) \to (13, 3)$
$(3, 2) \to (14, 1)$	$(7, 2) \to (14, 6)$	$(11, 2) \to (14, 11)$	$(15, 2) \to (14, 3)$
$(3, 3) \to (15, 1)$	$(7, 3) \to (15, 6)$	$(11, 3) \to (15, 11)$	$(15, 3) \to (15, 3)$
$(3, 4) \to (12, 0)$	$(7, 4) \to (12, 5)$	$(11, 4) \to (12, 10)$	$(15, 4) \to (12, 15)$
$(3, 5) \to (13, 0)$	$(7, 5) \to (13, 5)$	$(11, 5) \to (13, 10)$	$(15, 5) \to (13, 15)$
$(3, 6) \to (14, 0)$	$(7, 6) \to (14, 5)$	$(11, 6) \to (14, 10)$	$(15, 6) \to (14, 15)$
$(3, 7) \to (15, 0)$	$(7, 7) \to (15, 5)$	$(11, 7) \to (15, 10)$	$(15, 7) \to (15, 15)$
$(3, 8) \to (12, 4)$	$(7, 8) \to (12, 9)$	$(11, 8) \to (12, 14)$	$(15, 8) \to (12, 12)$
$(3, 9) \to (13, 4)$	$(7, 9) \to (13, 9)$	$(11, 9) \to (13, 14)$	$(15, 9) \to (13, 12)$
$(3, 10) \to (14, 4)$	$(7, 10) \to (14, 9)$	$(11, 10) \to (14, 14)$	$(15, 10) \to (14, 12)$
$(3, 11) \to (15, 4)$	$(7, 11) \to (15, 9)$	$(11, 11) \to (15, 14)$	$(15, 11) \to (15, 12)$
$(3, 12) \to (12, 13)$	$(7, 12) \to (12, 8)$	$(11, 12) \to (12, 2)$	$(15, 12) \to (12, 7)$
$(3, 13) \to (13, 13)$	$(7, 13) \to (13, 8)$	$(11, 13) \to (13, 2)$	$(15, 13) \to (13, 7)$
$(3, 14) \to (14, 13)$	$(7, 14) \to (14, 8)$	$(11, 14) \to (14, 2)$	$(15, 14) \to (14, 7)$
$(3, 15) \to (15, 13)$	$(7, 15) \to (15, 8)$	$(11, 15) \to (15, 2)$	$(15, 15) \to (15, 7)$

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